An elliptic pencil with the origin and a point on the real axis at a distance e as limiting points may be represented by

$$(x-e)^2+y^2-\lambda(x^2+y^2)=0$$

or

$$\left(x-\frac{e}{1-\lambda}\right)^2+y^2=\frac{\lambda e^2}{(1-\lambda)^2}, \tag{3}$$

for all real positive values of λ. For a definite λ, the distances of the limiting points from the center of the circle (3) are

$$\frac{e}{1-\lambda} \quad \text{and} \quad \frac{e}{1-\lambda}-e=\frac{\lambda e}{1-\lambda}.$$

The product of the two distances, $\lambda e^2/(1-\lambda)^2$, equals the square of the radius of the circle (3). From this it follows immediately that *the limiting points of an elliptic pencil are reflexions on every circle of the pencil* and we have

THEOREM 2. *A reflexion on any circle of an elliptic pencil transforms the pencil into itself. The same is, of course, also true of an hyperbolic and a parabolic pencil, since all circles of a pencil pass through two fixed points, which are imaginary in an elliptic, real and distinct in an hyperbolic and coincident in a parabolic pencil.*

In the circle (3) the distances of the extremities of the diameter on the axis from the origin, one of the limiting points, are

$$\frac{e}{1-\lambda}+\frac{e\sqrt{\lambda}}{1-\lambda}=\frac{e}{1-\lambda}(1+\sqrt{\lambda}),$$

and

$$\frac{e}{1-\lambda}-\frac{e\sqrt{\lambda}}{1-\lambda}=\frac{e}{1-\lambda}(1-\sqrt{\lambda}).$$

Inverting this we get for the center of the reflected circle (3) on the unit-circle around the origin the distance

$$\frac{1}{2}\left\{\frac{1-\lambda}{e(1+\sqrt{\lambda})}+\frac{1-\lambda}{e(1-\sqrt{\lambda})}\right\}=\frac{1}{e},$$

an expression which is independent of λ. Hence

THEOREM 3. *The reflexion of an elliptic pencil on a circle having one of the limiting points as a center is a concentric pencil with the reflexion of the other limiting point as a center.*[1]

§ 3. Circular Transformations.[2] **Groups of Substitutions.** A linear transformation of a complex variable

[1] A great number of other theorems on reflexion may be found in J. CASEY: *A treatise on Analytic Geometry*, chapt. III (1893); R. STURM: *Die Lehre von den geometrischen Verwandtschaften*, Vol. IV, pp. 72–95 (1909); W. FIEDLER: *Cyklographie* (1882).

[2] A comprehensive study of such transformations may be found in an article by F. N. Cole on "The linear functions of a complex variable," *Annals of Mathematics*, Vol. V, pp. 121–176 (1890).

$$z' = \frac{az + b}{cz + d}, \tag{4}$$

has the well-known property that it transforms circles into circles (including straight lines) and is therefore frequently called a circular transformation. If we transform the point z' by another circular transformation

$$z'' = \frac{a_1 z' + b_1}{c_1 z' + d_1}, \tag{5}$$

then the relation between z'' and z is given by

$$z'' = \frac{(aa_1 + cb_1)z + (ba_1 + db_1)}{(ac_1 + cd_1)z + (bc_1 + dd_1)}, \tag{6}$$

that is, z'' results from z by another circular transformation. Denoting the first substitution by which z is replaced by z' symbolically by S, similarly the second, by which z' passes into z'', by T and the third by U, we may say that the product of the substitutions S and T is U, or that S followed by T gives U. This may be symbolically written:

$$ST = U.$$

If some other substitution R carries the point z'' into z''', then there is again a linear substitution (geometrically a circular transformation) which changes z directly into z'''. This substitution may be symbolically written in the form

$$STR = U.$$

In this combination of substitutions the associative law holds, that is,

$$S(TR) = (ST)R,$$

while, in general, the commutative law is not true; namely

$$ST \neq TS.$$

If S is given, then by the inverse S^{-1} of S we understand the substitution of

$$\frac{dz - b}{-cz + a}$$

for z, which is obtained by solving

$$z = \frac{az' + b}{cz' + d}$$

or z'. It is easily seen that SS^{-1} is a substitution which leaves z invariant and is therefore called the identical substitution and may be indicated by

$$SS^{-1} = 1.$$

The foregoing definitions are sufficient to convey the idea of the group. For the sake of brevity this discussion is confined to finite groups.

A finite group of substitutions consists of a finite number of substitutions such that the product of any two of these is again one of the substitutions. The inverse of any substitution of the group belongs to the group. The group contains the identical substitution.

Such a group is obtained, for example, if we take n substitutions each S with the property that

$$SSS \cdots S \equiv S^n = 1.$$

Now $1, S, S^2, S^3, \cdots, S^{n-1}$ form a finite group of order n, which is at the same time cyclic. The reason for the attribute "cyclic" is apparent from the arrangement of the substitutions of the group.

§ 4. **Involutory Transformations.** An involutory transformation, or simply an involution in a complex plane may be defined as

$$z' = \frac{az + b}{cz - a}. \tag{7}$$

From this we find

$$z = \frac{az' + b}{cz' - a}.$$

If we transform by (7) z into z' and by the same transformation

$$z'' = \frac{az' + b}{cz' - a},$$

z' into z'', we find $z'' = z$, that is, by repeating an involution twice in succession, the original point z is transformed back into itself. Denoting the involutory substitution (7) by T, we have therefore, according to § 3,

$$T^2 = 1, \tag{8}$$

so that $1, T$ are the substitutions of a finite group of order 2, which is of course cyclic.

We shall now prove

THEOREM 4. *An involution in a complex plane is an inversion with respect to a fixed circle and a fixed axis through its center.*[1]

For this purpose turn the point z and its involutoric $(az + b)/(cz - a)$ through an angle ϕ about the origin. To their new positions apply a translation α, so that after the combined motion the points will be in the positions

$$e^{i\phi}z + \alpha \quad \text{and} \quad e^{i\phi}\frac{az + b}{cz - a} + \alpha.$$

[1] In a different form this proposition is also stated and proved in HOLZMÜLLER: *Einführung in die Theorie der isogonalen Verwandtschaften*, pp. 43–46 (1882).

If it is possible to determine ϕ, α and a real positive quantity ρ such that

$$(9) \qquad (e^{i\phi}z + \alpha)\left(e^{i\phi}\frac{az+b}{cz-a} + \alpha\right) = \rho^2,$$

for all values of z, the proposition will be proved.

Equation (9) may be written in the form
$e^{i\phi}(e^{i\phi}a + c\alpha) + \{e^{i\phi}(e^{i\phi}b - a\alpha) + \alpha(e^{i\phi} + c\alpha) - c\rho^2\}z + \alpha(e^{i\phi} - a\alpha) + a\rho^2 = 0.$
We have therefore the conditions

$$e^{i\phi}a + c\alpha = 0,$$
$$e^{i\phi}(e^{i\phi}b - a\alpha) = c\rho^2,$$
$$\alpha(e^{i\phi}b - a\alpha) = -a\rho^2.$$

But the first of these is a consequence of the last two. From these is found

$$(10) \qquad \alpha = \pm\frac{a\rho}{\sqrt{a^2+bc}}, \qquad e^{i\phi} = \mp\frac{c\rho}{\sqrt{a^2+bc}}.$$

As the absolute value of $e^{i\phi}$ is 1, we find for ρ

$$(11) \qquad \rho = \left|\frac{\sqrt{a^2+bc}}{c}\right|,$$

which is easily recognized as half the distance between the double-points of the involution. From (9) and (10) it would appear that the problem has two solutions. To show that the inversion is unique take first in (9) and (10) for α and $e^{i\phi}$ the positive and negative sign and designate the corresponding values by α_1 and ϕ_1. Taking $\alpha = \alpha_1$ and $\phi = \phi_1$, then

$$e^{i\phi_1}\cdot z \quad \text{and} \quad e^{i\phi_1}\cdot\frac{az+b}{cz-a}$$

are inverse with respect to a circle having $-\alpha_1$ as a center, ρ as given by (11) as a radius, and the line l through $-\alpha_1$ parallel to the real axis as an axis. Rotating the figure of this inversion back through an angle $-\phi_1$ we obtain the original points z and $(az+b)/(cz-a)$ and $-\alpha_1$ is moved to a position $\beta = -\alpha_1 e^{i\phi}$. The axis l is moved to a position g through β. But $\beta = +(a/c)$ is the midpoint between the double-points. *Consequently* z and $(az+b)/(cz-a)$ *are inverse with respect to a circle through the double-points of the transformation as extremities of a diameter and with the line joining them as an axis.*

Taking for α the other sign $\alpha_2 = -\alpha_1$, and $\phi_2 = \phi_1 + \pi$, we get the same result.

§ 5. **General Circular Transformation of an Inversion and an Ordinary Reflexion.** It is well-known that the transformed of a group is an isomorphic

group, that is, a group which obeys the same formal laws as the original group. Hence if we transform the cyclic group of order 2

$$\left(z, \frac{az+b}{cz-a}\right)$$

by any circular transformation

$$z' = \frac{\alpha z + \beta}{\gamma z + \delta},$$

and designate the substitutions $(az+b)/(cz-a)$ and $(\alpha z+\beta)/\gamma z+\delta)$ by the symbols S and T respectively, the transformed group

$$(1, T^{-1}ST)$$

is also of order 2, and is consequently geometrically represented by an involution, or an inversion on a circle.

This result may be stated as

THEOREM 5. *A circular transformation of an inversion is again an inversion.* In connection with theorem (1) we deduce at once

THEOREM 6. *A circular transformation of a reflexion on a circle is again a reflexion on a circle, or on a straight line.*[1]

Consider now the group of order 2,

$$(z, 2\lambda - z),$$

where λ is a constant, and which geometrically is represented by an ordinary reflexion on a straight line parallel to the axis of imaginaries at a distance λ from the origin. The transformed of this group by any circular transformation is again of order 2 and is geometrically represented by a reflexion on the circle which is the transformed of the line $x = \lambda$ in the z-plane.

Thus we can state

THEOREM 7. *A circular transformation transforms a system of parallel equidistant straight lines into a parabolic pencil of circles, so that of any three consecutive circles C_1, C_2, C_3, we may consider C_1 and C_3 as inversions or reflexions on C_2.*

§ 6. **Geometric Properties of Ring-Shaped Domains Connected with Infinite Cyclic Groups of Linear Substitutions.** As is well known every linear substitution between two complex variables may be written in one of the two forms,

$$(12) \qquad \frac{x_1 - a}{x_1 - b} = q\frac{x-a}{x-b}, \qquad (13) \quad \frac{A}{x_1 - a} = \frac{A}{x-a} + 1,$$

where x_1 and x are the variables and a and b are the double-points. In (12) we have the loxodromic case when $|q| \leqq 1$, elliptic when $|q| = 1$. In (13) we have the parabolic substitution. I shall confine myself to the loxodromic and parabolic

[1] Corresponds to theorem 4, p. 244, Osgood, *loc. cit.*

cases. Repeating the substitutions (12) and (13) each λ times in succession we get

$$\frac{x_\lambda - a}{x_\lambda - b} = q^\lambda \frac{x - a}{x - b}, \tag{14}$$

$$\frac{A}{x_\lambda - a} = \frac{A}{x - a} + \lambda, \tag{15}$$

representing, when λ assumes all integral values between $-\infty$ and $+\infty$, cyclic groups. Consider first the loxodromic group, where $|q| \neq 1$. By the linear substitution

$$c \cdot \frac{x - a}{x - b} = z, \tag{16}$$

the x-plane is transformed into the z-plane, and the substitutions of the transformed group assume the simple form

$$z_\lambda = q^\lambda z. \tag{17}$$

As the fundamental domain of this group we choose the ring-shaped surface between the unit-circle and the concentric circle of radius $|q|$. The remaining domains are bounded by the pencil of concentric circles with radii $|q^\lambda|$, where $-\infty < \lambda < +\infty$. Since for any three consecutive circles C_λ, $C_{\lambda+1}$, $C_{\lambda+2}$ there is

$$|q^\lambda| \cdot |q^{\lambda+2}| = |q^{\lambda+1}|^2,$$

we conclude that C_λ and $C_{\lambda+2}$ are reflexions on $C_{\lambda+1}$. Transforming back into the x-plane the pencil (C_λ) is transformed into the elliptic pencil with a and b as limiting points. The transformed circles K_λ corresponding to those of C_λ form the boundaries of the domains belonging to the group (14). From theorem (6) we conclude immediately:

THEOREM 8. *Of any three consecutive circles K_λ, $K_{\lambda+1}$, $K_{\lambda+2}$ of the domains associated with the cyclic loxodromic group, K_λ and $K_{\lambda+2}$ are reflexions on $K_{\lambda+1}$.*

In case of the parabolic group we may transform it by the substitution

$$\frac{A}{x - a} + c = z \tag{18}$$

into the z-plane, so that we get

$$z_\lambda = z + \lambda. \tag{19}$$

As the fundamental region we choose the strip between the axis of imaginaries and a line parallel to it at a distance equal to unity. The domains are now bounded by a system of parallel equidistant lines. Hence according to theorem (7) in the corresponding pencil of the x-plane of any three consecutive circles K_λ, $K_{\lambda+1}$, $K_{\lambda+2}$ we find as in the foregoing theorem that K_λ and $K_{\lambda+2}$ are reflexions on $K_{\lambda+1}$. These properties which are well-known results[1] also follow from the theory of conformal transformations.

[1] See § 16, p. 200, Vol. 1, *Modulfunctionen*, by Klein-Fricke.

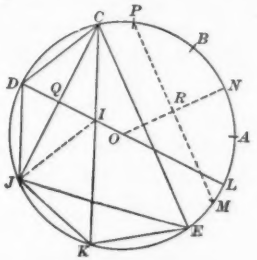
CONCERNING THE REGULAR INSCRIBED HEPTAGON.

By S. A. JOFFE, New York City.

Mr. J. Q. McNatt's article "A Geometrical Discussion of the Regular Inscribed Heptagon," which appeared in the January number of this MONTHLY, pages 13 and 14, contains a very interesting and ingenious method of arriving at what the author calls "The Heptagon Cubic," from the solution of which he shows that the square of the side of the regular heptagon equals $\frac{3}{4}$, as a very close approximation.

Without detracting from the value of the method, I should like to point out that some of the details in the process may be omitted or simplified, and it has occurred to me that it may be worth while to present here the method in an abbreviated form, thus making the matter more widely known. In preparing the following lines, I have endeavored to preserve as much of the original figure and of the author's text as was consistent with the object of simplification.

To calculate the length of the side of a regular inscribed heptagon in terms of the radius as a unit, suppose, in the accompanying figure, that the circumference is divided into seven equal parts at the points A, B, C, D, J, K, and E.



Draw the diameter DL and the chords JE, CJ, CK and CE, and let CJ and CK cut DL at Q and I respectively. Join I and J.

Let the length of the side of the regular inscribed heptagon be h units. We have $DL : DC :: DC : DQ$; hence $DQ = \frac{1}{2}h^2$. Moreover, $QC = \sqrt{DC^2 - DQ^2} = \sqrt{h^2 - \frac{1}{4}h^4} = \frac{1}{2}h\sqrt{4 - h^2}$. From the isosceles triangle DCI we have $CI = CD = h$, and $DI = 2DQ = h^2$; from the isosceles triangle CIJ we have $IJ = CI = h$, and $CJ = 2CQ = h\sqrt{4 - h^2}$, so that

$$JE = h\sqrt{4 - h^2}. \tag{1}$$

Now, by geometry $IL \times DI = CI \times IK$, and since $IL = DL - DI = 2 - h^2$, we have $(2 - h^2) \times h^2 = h \times IK$; hence

$$IK = 2h - h^3, \tag{2}$$

and since $CK = CI + IK = 3h - h^3$, and $CE = CK$, therefore

$$CE = 3h - h^3. \tag{3}$$

The two isosceles triangles CJE and IJK, having equal angles at E and K respectively, are similar. Hence we have $JE : CE :: JK : IK$, or, using (1), (2) and (3)

$$h\sqrt{4 - h^2} : (3h - h^3) :: h : (2h - h^3).$$

Hence

$$(2 - h^2)\sqrt{4 - h^2} = 3 - h^2. \tag{4'}$$

Squaring, we get

$$(4 - 4h^2 + h^4)(4 - h^2) = 9 - 6h^2 + h^4,$$

and expanding,

$$16 - 20h^2 + 8h^4 - h^6 = 9 - 6h^2 + h^4.$$

Finally, transposing and simplifying, we obtain the author's *Heptagon Cubic:*

$$7 - 14h^2 + 7h^4 - h^6 = 0. \tag{5}$$

Solving by Horner's method, we find $h^2 = .7530203962821\ldots = \frac{3}{4}$, approximately.

Remark. It will thus be seen that while there is introduced a new line IJ, we dispense with the consideration of the line OC, and with both the consideration and the computation of the author's lines SE, SJ, SK and SC. As a result, the equation (4') appears in a much simpler form than the author's equation (4).

The approximate construction of the heptagon may also be simplified as follows:

Let M, N and P be three consecutive vertices of an inscribed regular hexagon. Draw the chord MP and the radius ON, and let MP meet ON in R. Then MR is, approximately, the length h of the side of the regular inscribed heptagon. The reason is self-evident: approximately, $h = \frac{1}{2}\sqrt{3}$, and MP, as the side of a regular inscribed triangle, $= \sqrt{3}$, so that $MR = \frac{1}{2}\sqrt{3}$, and therefore $MR = h$, approximately.

A PROBLEM IN NUMBER THEORY.

By GEO. A. OSBORNE, Massachusetts Institute of Technology.

§ 1. When is the sum of the squares of two successive integers a perfect square? The following are examples:

$$3^2 + 4^2 = 5^2, \quad 20^2 + 21^2 = 29^2. \quad \text{The next is } 119^2 + 120^2 = 169^2.$$

The numbers 3, 20, 119, . . . are the terms of a series

$$0,\ 3,\ 20,\ 119,\ 696,\ \cdots\ u_n,\ u_{n+1}, \tag{1}$$

where

$$u_{n+1} = 6u_n - u_{n-1} + 2. \tag{2}$$

This may be proved as follows:

From (2), which is the relation between any three successive terms of (1), we may derive the relation between any two successive terms as follows:

From (2)

$$u_{n+1} + u_{n-1} = 6u_n + 2,$$

$$u_{n+1}^2 - u_{n-1}^2 = (6u_n + 2)(u_{n+1} - u_{n-1}),$$

$$(u_{n+1} - 1)^2 - 6u_n u_{n+1} = (u_{n-1} - 1)^2 - 6u_n u_{n-1},$$

Adding to each member $(u_n - 1)^2$, we have

$$(u_{n+1} - 1)^2 + (u_n - 1)^2 - 6u_n u_{n+1} = (u_n - 1)^2 + (u_{n-1} - 1)^2 - 6u_{n-1}u_n, \quad (3)$$

which is of the form $f(n) = f(n-1)$.

Hence by induction, $f(n) = c$, a constant independent of n. By applying the first member of (3) to the terms 3 and 20, we find $c = 5$. Hence

$$(u_{n+1} - 1)^2 + (u_n - 1)^2 - 6u_n u_{n+1} = 5 \quad (4)$$

is the relation between any two successive terms of (1).

Solving (4) with respect to u_{n+1}, we have

$$u_{n+1} = 3u_n + 1 \pm 2\sqrt{2u_n^2 + 2u_n + 1}, \quad (5)$$

in which the lower sign is rejected since, otherwise, the right member would be less than u_n. It follows from (5) that

$$2u_n^2 + 2u_n + 1 = \text{a square},$$

that is,

$$u_n^2 + (u_n + 1)^2 = \text{a square},$$

one part of the result which was to be proved.

§ 2. From the terms of (1) we may write

$$\begin{aligned}
0^2 + 1^2 &= 1^2,\\
3^2 + 4^2 &= 5^2,\\
20^2 + 21^2 &= 29^2,\\
119^2 + 120^2 &= 169^2,\\
696^2 + 697^2 &= 985^2,\\
4059^2 + 4060^2 &= 5741^2,\\
23660^2 + 23661^2 &= 33461^2,\\
137903^2 + 137904^2 &= 195025^2,\\
803760^2 + 803761^2 &= 1136689^2,\\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{aligned}$$

The second members are terms of a series,

$$1,\ 5,\ 29,\ \cdots u_n,\ u_{n+1},$$

where

$$u_{n+1} = 6u_n - u_{n-1}.$$

§ 3. It remains to be shown that the terms of the series (1) are the *only* integers that satisfy the condition

$$N^2 + (N+1)^2 = \text{a square.} \tag{6}$$

If from (4) we express u_n in terms of u_{n+1}, we have

$$u_n = 3u_{n+1} + 1 - 2\sqrt{2u_{n+1}^2 + 2u_{n+1} + 1}, \tag{7}$$

from which

$$u_{n-1} = 3u_n + 1 - 2\sqrt{2u_n^2 + 2u_n + 1}. \tag{8}$$

Consider the equation

$$y = 3x + 1 - 2\sqrt{2x^2 + 2x + 1}. \tag{9}$$

Then

$$2y^2 + 2y + 1 = 2(3x + 1 - 2\sqrt{2x^2 + 2x + 1})^2 + 2(3x + 1 - 2\sqrt{2x^2 + 2x + 1}) + 1$$

$$= (4x + 2 - 3\sqrt{2x^2 + 2x + 1})^2.$$

Hence if

$$2x^2 + 2x + 1 = \text{a square,}$$

then also

$$2y^2 + 2y + 1 = \text{a square.}$$

That is, if x satisfies (6), so does y. Comparing (9) with (7) and (8), it appears that if $x = u_{n+1}$, $y = u_n$; and if $x = u_n$, $y = u_{n-1}$. And as y is an increasing function of x, since

$$\frac{dy}{dx} = 3 - \frac{4x + 2}{\sqrt{2x^2 + 2x + 1}} > 0,$$

it follows that if $u_n < x < u_{n+1}$, then $u_{n-1} < y < u_n$. That is, if there is an integer satisfying (6) between u_n and u_{n+1}, there is another such integer between u_{n-1} and u_n.

There is no integer satisfying (6) between the terms 3 and 20; hence there is none between 20 and 119, and consequently none between any two successive terms of the series (1).

§ 4. In the list of equations in § 2, it may be noticed that none of the successive integers end in 2, 5 or 8. As the final digits recur, it follows that

The sum of the squares of two successive integers, one of which ends in 2, 5 *or* 8, *cannot be a perfect square.*

NOTE ON PERFECT MAGIC SQUARES FOR 1914.

By B. L. REMICK, Manhattan, Kansas.

In the February MONTHLY, Mr. V. M. Spunar states that the only values of n in Zerr's formula yielding positive elements for the 1914 magic square are 3 and 11. These are indeed the only "prime" values of n. He has, however, overlooked $n = 4$ and $n = 12$, which yield the following squares with elements all positive:

$n = 4$

471	485	484	474
482	476	477	479
478	480	481	475
483	473	472	486

$n = 12$

231	89	229	91	227	93	94	224	96	222	98	220
100	218	102	216	104	214	213	107	211	109	209	111
207	113	205	115	203	117	118	200	120	198	122	196
124	194	126	192	128	190	189	131	187	133	185	135
183	137	181	139	179	141	142	176	144	174	146	172
148	170	150	168	152	166	165	155	163	157	161	159
160	158	162	156	164	154	153	167	151	169	149	171
147	173	145	175	143	177	178	140	180	138	182	136
184	134	186	132	188	130	129	191	127	193	125	195
123	197	121	199	119	201	202	116	204	114	206	112
208	110	210	108	212	106	105	215	103	217	101	219
99	221	97	223	95	225	226	92	228	90	230	88

BOOK REVIEWS.

A History of Japanese Mathematics. By DAVID EUGENE SMITH and YOSHIO MIKAMI. The Open Court Publishing Company, Chicago, 1914. vii + 288 pages. $3.50.

This book is a beautiful specimen of the printer's art. The paper, the type, and the illustrations make it a work which it is a delight to read almost aside from the text. I hasten to add that text and content are in harmony with the dress. The sympathetic portrayal of the development of Japanese mathematics, largely indigenous and as the authors well state, "like her art, exquisite rather than grand," will appeal to a wide circle of readers and will contribute to a juster and broader appreciation of the Japanese genius.

The history of Japanese mathematics is divided into six periods. The first extends to 552 A. D. and was almost entirely a native development. The second period, from 552 to 1600, was characterized by the predominance of Chinese mathematics. The third period was a kind of Renaissance culminating in the appearance of Seki, the most famous Japanese mathematician. The fourth period, 1675 to 1775, and the fifth, 1775 to 1868, are marked by the complete development of the native mathematics, the *wasan*, in the second of these two periods somewhat influenced by European mathematics. The sixth is the period of the present, the mathematics of the world which knows nothing of political and racial boundaries.

The earliest and second periods are treated in 17 pages which is significant of the fact that little is known definitely of the history of these times. The development of the *soroban* from the Chinese *swan-p'an* and the methods of operating with this instrument are given in a detailed and satisfactory manner. Incidentally it seems worthy of note that *swan-p'an* with the meaning in Chinese of "reckoning table" corresponds to the Greek word from which we have "abacus," also having the meaning "table," particularly for bankers. The "sangi" or numeral rods, another type of palpable arithmetic, are explained both as used for representing numbers and also as applied to algebra. Numerical approximation of the roots of equations, especially in connection with the circle, made a particular appeal to the Japanese as well as to the Chinese. Magic squares and even magic circles received much attention in the third period. Three chapters are given over to Seki Kōwa, 1642–1708, and his pupils, including an extended discussion of the "yenri" or circle principle, an approach to the methods of the calculus. The remaining six chapters are devoted respectively to the eighteenth century, Ajima Chokuyen, the opening of the nineteenth century, Wada Nei, the close of the old Wasan, and the introduction of occidental mathematics. An index and vocabulary add to the usefulness of the work.

The typographical errors and other slips which I have noted are few in number and unimportant: p. 15 "compotus rolls," probably better "computus rolls"; p. 19, n. 6, "I" for "we"; p. 29, "aboud"; p. 31, "nstrument"; p. 51, "Latin *res* and the Italian *cosa*, both of which had undoubtedly come from the

East," but we know that these are direct translations from the Arabic *shai*, meaning *thing*; p. 51, n. 4, Egyptian *hau* should be *ahau* for *heap;* p. 80, "Arabic numerals," elsewhere "Hindu-Arabic numerals"; p. 279, "develpoed."

In every way this work can be commended to the student of the history of science and to the student of Japanese civilization, and quite as much to the general reader, for the greater part of the story is not at all technical. May this beautiful product of a German printer, W. Drugulin, Leipzig, put out by an American publishing house, the joint work of a Japanese and an American, be symbolical of a better mutual understanding between these countries.

L. C. KARPINSKI.

The Development of Mathematics in China and Japan; Abhandlungen zur Geschichte der mathematischen Wissenschaften, Vol. XXX. By YOSHIO MIKAMI. Teubner, Leipzig, 1913. G. E. Stechert and Co., New York. x + 347 pages. $5.50 net.

In 1910 Mr. Mikami published in the same series as this volume the work, *Mathematical Papers from the Far East,* which attracted favorable attention. The present work is of great importance because it brings systematized information about the history of mathematics in China and Japan, based upon a study of such original documents as remain. Regret must be expressed that the book is marred, even as the preceding work, by faulty English. The prefatory note by G. B. Halsted implies that the task of correcting the English was entrusted to him, but instead of correcting he says: "This is not the idiom of England nor of the United States nor have I striven so to cramp it." I take it that Mr. Mikami desired to bring out the work in idiomatic English and censure for the many errors of construction as well as the actually unintelligible statements can be attached only to the American scholar to whom the revision was entrusted. I think it is also fair to say that no American publishing house would bring out a work in the German language with such faults in usage, spelling, and construction as disfigure the work under consideration. The reviewer makes no attempt to list these errors.

There is no better evidence of the uncertainty which attaches to the ancient mathematics of China than the fact that Mr. Mikami's description of the *Arithmetic in Nine Sections* is entirely different (footnote, p. 10) from the description given by T. Hayashi in his *Brief History of Japanese Mathematics* (Nieuw Archief, Tweede Reeks, Deel VI, 306–307, not accessible to me). However this uncertainty is also characteristic of the history of ancient Hindu mathematics.

One of the most striking chapters of the work (Chap. IV) deals with *The Arithmetical Classic of Sun-Tsu.* The operations of multiplication and division and extraction of square root correspond almost precisely, mechanically, to these operations as taught in the early treatises explaining the Hindu art of reckoning. Thus in multiplication the unit of the lower number is placed below the highest digit of the upper number and the highest digit of the upper number is then multiplied by the lower number, the product being arranged in a line between the

two. Commonly the partial products in the treatises based upon the arithmetic of Al-Khowarizmi are placed above the two lines of multiplier and multiplicand, or on the upper line, this number being deleted in the course of the operation. Then the lower number is drawn back one digit and the work proceeds as before both in the Chinese work and in the European, which is based upon the Arabic and Hindu. The correspondence seems too striking to be wholly accidental but the connection has not yet been established. These calculations were effected doubtless with the *sangis* or calculating pieces upon some sort of abacus. The date of the treatise is left undetermined. The mathematician, Tai Cheng (1722–1777), maintains that the composition of Sun-Tsu's treatise could not ante-date the reign of the Han emperor, Ming-Ti (first century A.D.). Apparently Mr. Mikami construes this to mean that this work was composed about the first century of the Christian era, but no better evidence than the uncertain statement I have quoted is presented as to the age of the treatise. Somewhat similar treatises on the fundamental operations are found in the sixth century in China.

The Tsu Ch'ung-chih (fifth century) approximation for the value of π, namely 355/113, which is easily remembered from the sequence 1 1 3 3 5 5, should be familiar to every teacher of elementary mathematics and this might well be designated as the Chinese approximation if the name "Tsu Ch'ung-chih's approximation" seems too formidable. Other points particularly worthy of note include the *t'ai-yen* method for the solution of indeterminate equations, the *celestial element* process for the treatment of numerical equations, "Pascal's triangle" in China in the early fourteenth century, and Seki's development of the determinant idea in the seventeenth century. One *t'ai-yen* problem proposed by Sun-Tsu is to find a number which divided by 3, 5, and 7 respectively gives as remainders 2, 3, and 2 respectively. This is of particular interest because the same problem with the exception of the third remainder, 4 replacing 2, given by Leonard of Pisa in 1202, bears witness to contact between East and West. The thirteenth century studies by Ch'in Chiu-shao on approximation of the roots of numerical equations correspond closely to the methods of Newton and Horner.

Numerical errors are somewhat common in the work, as for example: 293 for 233 (p. 32); 30 9/60 for 36 9/10 (p. 54); 1337 1/20 for 397 3/4 (p. 54). The problem of the 10 foot bamboo which is broken at such a point that it touches the ground 3 feet from the stem (p. 23) is not found in Brahmagupta's work but twice in Bhaskara (Colebrooke, *Algebra . . . from the Sanskrit of Brahmagupta and Bhâscara*, London, 1817, pp. 203–204 and 64–65). There appears to be no point in stating that there is "no use of decimals" (p. 12) in these ancient works, for decimal subdivisions abound and decimal fractions as we have them, using the decimal point, would be most unlikely to occur. The explanation (p. 2) of the *chia-tsu* or sexagesimal system of numeration is not at all clear, and the notation requires ten numeral words in the first series whereas only nine are given.

Mikami compares Seki with Newton (p. 158). He speaks of "the Japanese Newton," and says "If Seki did not surpass Newton in his achievements, yet he was no inferior of the two" (Galileo being the other to whom this statement

refers). The reader will look in vain in Mr. Mikami's work for any justification of this effusive praise which flatters neither Seki, nor Newton, nor the Japanese. Seki was undoubtedly for his time and place an able mathematician, but the luster of his light is only dimmed by comparison with Newton or Galileo.

In concluding our review we wish to state again that Mr. Mikami has rendered a real service to the history of science by his exposition of the development of mathematics in China and Japan.

L. C. KARPINSKI.

PROBLEMS AND SOLUTIONS.

B. F. FINKEL, CHAIRMAN OF THE COMMITTEE.

PROBLEMS FOR SOLUTION.

SPECIAL NOTICE. In proposing problems and in preparing solutions, contributors will please follow the form established by the MONTHLY, as indicated on the following pages.

In particular, a solution should be preceded by the number of the problem, the name and address of the proposer, the statement of the problem, and the name and address of the solver.

The solution should then be given with careful attention to legibility, accuracy, brevity without obscurity, paragraphing and spacing, having in mind the form in which it will appear on the printed page.

Please use paper of letter size, write on one side only, leaving ample margins, put one solution only on a single sheet and include only such matter as is intended for publication.

Drawings must be made *clearly* and *accurately* and an extra copy furnished on a *separate sheet* ready for the engraver.

Unless these directions are observed by contributors, solutions must be entirely rewritten by the committee or else rejected.

Selections for this department are made two months in advance of publication.

Please send all solutions direct to the chairman of the committee.

MANAGING EDITOR.

ALGEBRA.

When this issue was made up, solutions had been received for 403–4–5–7–9 and 410. Solutions of 406 and 408 are desired.

413. Proposed by C. N. SCHMALL, New York City.

Apply Euler's transformation to show that

$$1 + 2^2x + 3^2x^2 + 4^2x^3 + 5^2x^4 + \cdots = \frac{1+x}{(1-x)^3}.$$

(BROMWICH, *Theory of Infinite Series*, p. 62, ex. 20.)

414. Proposed by R. D. CARMICHAEL, Indiana University.

Prove by means of an example that one of the series

$$\sum_{k=1}^{\infty} \frac{1}{c_k}, \qquad \sum_{k=1}^{\infty} \frac{1}{c_k - 1}, \qquad c_k \neq 0, 1,$$

may be divergent while the other is convergent.

415. Proposed by C. N. SCHMALL, New York City.

Show that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}.$$

(BROMWICH, *Infinite Series*, p. 187; and also p. 452, ex. 7, (iii)).

416. Proposed by C. E. FLANAGAN, Wheeling, Va.

The sides of a given rectangle are a and b, in which a rectangle is to be inscribed one of whose sides is c. Find the other side, using Euler's rule for quartics.

417. Proposed by A. J. RICHARDSON, Marquette, Mich.

Required to reduce the quartic

$$x^4 + px^2 + qx + r = 0$$

to the form

$$(x^2 + k)^2 = [(2k - p)x + (2k - q)]^2,$$

wherein k is the solution of a certain cubic. Hence, express the solution of the given quartic in terms of p, q, r, and k.

GEOMETRY.

When this issue was made up, solutions had been received for 434–5–6 and 439. Solutions of 432–3 and 441–2–3 are desired.

441. Proposed by H. E. TREFETHEN, Colby College.

In the triangle ABC find the locus of all points at which the sides AB, AC subtend equal angles.

442. Proposed by J. B. SMITH, Hampden-Sidney, Va.

If any three straight lines, AD, BE, CF, be drawn from the corners of the triangle ABC to the opposite sides, a, b, c, they will enclose an area. If Δ, Δ'' be the areas of the triangles ABC, DEF show that

$$\frac{\Delta''}{\Delta} = \frac{(AF \cdot BD \cdot CE - AE \cdot CD \cdot BF)^2}{(ab - CE \cdot CD)(bc - AE \cdot AF)(ca - BF \cdot BD)},$$

where the signs of the factors are to be determined by the following rule: Each segment being measured from one of the corners of the triangle ABC, along one of the sides, is to be regarded as positive or negative according as it is drawn towards or from the other corner in that side.

443. Proposed by C. N. SCHMALL, New York City.

A quadrilateral of any shape whatever is divided by a transversal into two quadrilaterals. The diagonals of the original figure and those of the two resulting (smaller) figures are then drawn. Show that their three points of intersection are collinear.

CALCULUS.

When this issue was made up, solutions had been received for 346–7–9 and 351–4–6–7. Solutions of 332–5, 340–5–8 and 352–3–5 are desired. A complete solution of 339 is also desired.

363. Proposed by B. F. FINKEL, Drury College.

The axis of a right prism whose cross-section is a regular polygon of n sides coincides with the diameter of a sphere of radius R. Find the surface of the sphere included within the prism.

364. Proposed by EMMA GIBSON, Drury College.

Solve the differential equation

$$(xp - y)^2 = a(1 + p^2)(x^2 + y^2)^{3/2}, \qquad \text{where } p = dy/dx.$$

365. Proposed by C. N. SCHMALL, New York City.

Show that the area inclosed by each of the following three curves is equal to that of the circle of radius a; viz., πa^2.

(1) $$a^2x^2 = y^2(2a - y),$$

(2) $$a^2 - x^2 = (y - mx^2)^2,$$

(3) $$(xy + c + bx^2)^2 = x^2(a^2 - x^2).$$

MECHANICS.

When this issue was made up, solutions had been received for 285–8–9 and 292. Solutions of 266, 268, 269, 271 to 275, inclusive, and 277 to 279 inclusive are desired.

293. Proposed by B. F. FINKEL, Drury College.

A man of weight W stands on smooth ice; prove that if, when he gradually parts his legs, kept straight, with his feet in contact with the ice, the pressure of his feet on the ice be constant, his head will descend with uniform acceleration; and that, if f be the acceleration of his head, when his feet exert no pressure on the ice, their pressure on the ice, were f' the acceleration of his head, would be equal to $\frac{f-f'}{f}W$.

(From WALTON'S *Problems in Theoretical Mechanics.*)

294. Proposed by EMMA GIBSON, Drury College.

A sphere, revolving about a diameter and not acted on by any extraneous force, expands symmetrically; prove that its vis viva varies inversely as its moment of inertia about a diameter.

NUMBER THEORY.

When this issue was made up, solutions had been received for 206–7, 210, 213. Solutions of 192 and 196 are desired.

215. Proposed by R. D. CARMICHAEL, Indiana University.

Find one or more values of n such that a polygon of n sides shall have the number of its diagonals equal to the cube of an integer.

216. Proposed by ELIJAH SWIFT, Princeton, N. J.

If p is a prime > 3, show that $\Sigma_{a=1}^{a=p-1} 1/a \equiv 0 \pmod{p^2}$, where $1/a = x$, if $ax \equiv 1 \pmod{p^2}$

217. Proposed by E. T. BELL, Seattle, Wash.

(i) If r is a prime greater than 2, and $p \equiv 2^n r + 1$ is prime, the only solution, when n is greater than 2, of $x^n - y^n = p$, is $n = 3$, $x = 2$, $y = 1$.

(ii) The only primes that are simultaneously of the forms $4k + 1$ and $3^m - 2^m$ are 1 and 5.

(iii) Generalize (ii).

SOLUTIONS OF PROBLEMS.

ALGEBRA.

385. Proposed by J. F. LAWRENCE, Stillwater, Oklahoma.

Show that if p is prime and > 3, $(2p)! - 2 \cdot p!p!$ is divisible by p^3.

SOLUTION BY ELIJAH SWIFT, Princeton, N. J.

Dividing the above expression by $2p \cdot p!$ we have to show that

$$(1) \qquad (2p-1)(2p-2)(2p-3)\cdots(2p-(p-1)) - (p-1)!$$

is divisible by p^3.

Consider the polynomial $P(x) = (x-1)(x-2)(x-3)\cdots(x-p+1)$.[1] If this is expanded in powers of x, we obtain an expression of the form

$$x^{p-1} - A_1 x^{p-2} + A_2 x^{p-3} + \cdots + A_{p-3}x^2 - A_{p-2}x + A_{p-1},$$

[1] See BACHMANN, *Niedere Zahlentheorie*, p. 155. I have changed signs, but evidently the same results hold.

where

$$A_1 = 1 + 2 + 3 + \cdots + p - 1$$
$$A_2 = 1\cdot 2 + 1\cdot 3 + \cdots + 2\cdot 3 + \cdots + (p-2)(p-1)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$A_{p-1} = 1\cdot 2\cdot 3\cdots(p-1).$$

It is proved by Bachmann that each of these A's, except A_{p-1}, is divisible by p.

Obviously, from the expression of $P(x)$ in factor form $P(p) = (p-1)!$ It follows that $P(p) - (p-1)! = p^{p-1} - A_1 p^{p-2} + \cdots + A_{p-3}p^2 - A_{p-2}p = 0$. If then A_{p-3} is divisible by p, A_{p-2} will be divisible by p^2. If $p = 3$, however, $A_{p-3} = 1$, or rather there is no A_{p-3}, and our reasoning breaks down.

Now expand (1). We shall evidently have $P(2p) - (p-1)! = (2p)^{p-1} - A_1(2p)^{p-2} + \cdots + A_{p-3}(2p)^2 - A_{p-2}(2p)$. But A_{p-3} is divisible by p, and A_{p-2} by p^2. Therefore, (1) is divisible by p^3, and $(2p)! - 2\cdot p!p!$ by p^5.

398. Proposed by R. D. CARMICHAEL, Indiana University.

In the equation $x^3 + \alpha x + \beta = 0$, α is an integer divisible by p^3 and β is an integer divisible by p, p being a prime number. Prove that β is divisible by p^3 if the equation is reducible.

SOLUTION BY ELMER SCHUYLER, Brooklyn, New York.

Let $(x^2 + lx + m)(x + n) \equiv x^3 + \alpha x + \beta$. Then since $\alpha = p^2t$ and $\beta = pr$, $l = -n$; $m - n^2 = p^2t$; $mn = pr$. Hence, $m = pr/n$ and $pr/n - n^2 = p^2t$, or $pr - n^3 = p^2nt$. Consequently, n is divisible by p and $n = ps$, since n clearly is an integer. Whence $pr - p^3s^3 = p^3st$ or $r/p^2 = st - s^3$. Hence, we have $\beta = p^3(st - s^3)$.

Also solved by B. LIBBY, G. W. HARTWELL, A. M. HARDING, and ELIJAH SWIFT.

399. Proposed by W. H. BUSSEY, University of Minnesota.

A borrows from B \$1,500 and pays back \$34 a month for 63 months. If the last payment closes the account, what rate of interest has A been paying.

SOLUTION BY GEO. W. HARTWELL, Hamline University.

This problem is equivalent to the following: B pays A \$1,500 for an annuity of \$34 per month to run 63 months. What is the rate realized?

Let r = the annual rate. Then $r/12$ = the monthly rate. Hence,

$$34\left(1+\frac{r}{12}\right)^{62} + 34\left(1+\frac{r}{12}\right)^{61} + \cdots + 34 \equiv 34\left[\frac{\left(1+\frac{r}{12}\right)^{63} - 1}{\frac{r}{12}}\right] = 1{,}500,$$

or the total number of dollars realized.

Solving this equation by the usual method, we find $r = 14.31$ per cent.

Also solved by G. Y. SASNOW, ALBERT R. NAUER, WILLIAM CULLUM, W. C. EELLS, and CLIFFORD N. MILLS.

400. Proposed by C. N. SCHMALL, New York City.

Sum the series $1 + 2x + 3x^2 + 4x^3 + \cdots$.

(BROMWICH, *Infinite Series*, p. 129, ex. 1.)

I. Solution by Geo. W. Hartwell, Hamline University.

Let

$$S = 1 + 2x + 3x^2 + 4x^3 + \cdots.$$

Then

$$(1 - x)S = 1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1 - x}, \text{ for } -1 < x < +1.$$

Hence

$$S = \frac{1}{(1 - x)^2}.$$

II. Solution by Elijah Swift, Princeton University.

The series converges for $-1 < x < +1$ and may be integrated in the open interval -1 to $+1$.

Then $\int_0^x S(x)dx = x + x^2 + x^3 + \cdots$, which is equal to $\frac{x}{1 - x}$. Differentiating this, we obtain $\frac{1}{(1 - x)^2}$ as the sum of the given series.

III. Solution by J. Brooks Smith, Hampden Sidney, Va.

The following solution is given by Charles Smith, *Treatise on Algebra*, ex. 1, p. 413.

$$S_{n+1} = 1 + 2x + 3x^2 + 4x^3 + \cdots + (n + 1)x^n,$$

$$(1 - x)^2 = 1 - 2x + x^2.$$

Hence,

$$(1 - x)^2 \cdot S_{n+1} = 1 + x^{n+1}[n - 2(n + 1)] + (n + 1)x^{n+2}$$

(all the other terms vanishing on account of the identity,

$$k - 2(k - 1) + k - 2 \equiv 0).$$

Hence

$$(1 - x)^2 S_{n+1} = 1 - (n + 2)x^{n+1} + (n + 1)x^{n+2};$$

whence

$$S_{n+1} = \frac{1}{(1 - x)^2} - \frac{(n + 2)x^{n+1} - (n + 1)x^{n+2}}{(1 - x)^2}.$$

When $n \doteq \infty$, $S_{n+1} \doteq S \doteq \frac{1}{(1 - x)^2}$ for $-1 < x < +1$.

Solved in various other ways by Elmer Schuyler, B. Libby, Horace Olson, F. M. Morgan, A. L. McCarty, C. Hornung, A. M. Harding, Clifford N. Mills, H. C. Feemster, Oscar Schmiedel, and A. G. Caris.

A solution of 396 was received from Elijah Swift too late for credit in the last issue.

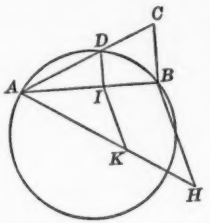
GEOMETRY.

428 Proposed by R. D. Carmichael, Indiana University.

On a given chord of a circle as a base construct a right triangle with vertex outside of the circle such that its hypotenuse shall be bisected by its point of intersection with the circle. Are ruler and compasses sufficient to construct a triangle whose hypotenuse shall be thus divided in any ratio whatever?

SOLUTION BY EMMA M. GIBSON, Drury College.

The ruler and compasses are sufficient to construct a triangle whose hypotenuse shall be divided in any ratio as $m : n$, according to the conditions of the problem.



For let AB be the given chord. From A draw any line as AH and from A lay off $AK = m$ and $KH = n$. Then connect B and H and draw KI parallel to HB and intersecting AB in I. At I erect a perpendicular to AB intersecting the circle in D. Join A and D by a straight line and produce it until it meets a perpendicular erected at B in C. Then the right triangle ABC has its hypotenuse, AC, divided at D in the ratio of $m : n$.

For in the similar triangles AIK and ABH, $AI : IB = m : n$ and in the similar triangles ABC and AID, $AI : IB = AD : DC$. Hence, $AD : DC = m : n$.

When $m = n$, D bisects the hypotenuse.

Also solved by CLIFFORD N. MILLS, F. M. MORGAN, C. HORNUNG, HORACE OLSON, and S. W. REAVES.

429. Proposed by JOHN A. BIGBEE, Little Rock, Ark.

In the trihedral angle V-ABC, the face angle AVB is bisected by the straight line VD. Is it true that the angle DVC is less than, equal to, or greater than, half the sum of the angles AVC and BVC, according as $\angle CVD$ is less than, equal to, or greater than 90°?

SOLUTION BY A. M. HARDING, University of Arkansas.

(1) $\angle CVD < 90°$. (Fig. 1.) Take $VA = VB$. Join A and B. Let this line cut the bisector of $\angle AVB$ at D. Through D pass a plane perpendicular to

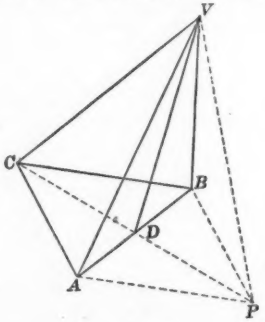


FIG. 1.

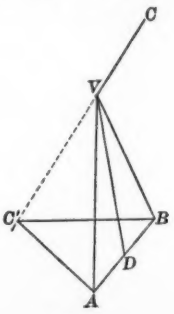


FIG. 2.

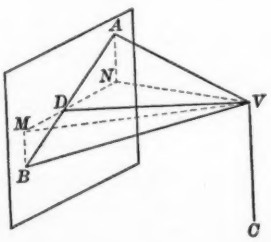


FIG. 3.

VD cutting VC at C. Produce CD to P making $DC = DP$. Draw the lines AP, BP, and VP. From this construction it easily follows that $\angle BVC = \angle PVA$ and $\angle CVP = 2\angle CVD$. But $\angle CVP < \angle AVC + \angle PVA$. Hence $\angle CVD < \frac{1}{2}(\angle AVC + \angle BVC)$.

(2) $\angle CVD > 90°$. (Fig. 2.) Produce CV through V and draw plane ABC' as in (1). Then $2\angle C'VD < \angle AVC' + \angle BVC'$ by (1). Hence $2(180° - \angle C'VD) > 180° - \angle AVC' + 180° - \angle BVC'$ or $\angle CVD > \frac{1}{2}(\angle AVC + \angle BVC)$.

(3) $\angle CVD = 90°$. (Fig. 3.) In this case the plane through AB is parallel to VC. Draw a plane MNV through VD perpendicular to VC and cut by planes BVC and AVC in the lines MV and NV respectively. It can be easily shown that $\angle MVB = \angle NVA$. Hence

$$\angle AVC + \angle BVC = 180°$$

and

$$\angle CVD = \tfrac{1}{2}(\angle AVC + \angle BVC).$$

Also solved by B. LIBBY, F. M. MORGAN, and GEO. W. HARTWELL.

CALCULUS.

338. Proposed by RICHARD LOCHNER, Philadelphia, Pa.

An elliptical field has a major axis of 100 feet and a minor axis of 10 feet. A cow is tethered at the end of the major axis and another at the end of the minor axis. If each cow can graze over half the field, how long is the rope of each? What is the area of the portion over which the cows can graze in common?

SOLUTION BY B. F. FINKEL, Drury College.

The central equation of the elliptic field is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Let r_1 be the length of rope by which one cow is tethered at the point A_1, the right-hand extremity of the major axis. The equation of the circle over which this cow can browse is $(x - a)^2 + y^2 = r^2$. The coördinates of the point of intersection, P_1, of this circle with the ellipse are

$$(x_1, y_1) \equiv \left(a\frac{[a^2 - \sqrt{b^4 + r^2(a^2 - b^2)}]}{(a^2 - b^2)}, \right.$$

$$\left. \sqrt{2a^4\frac{\sqrt{b^4 + r^2(a^2 - b^2)} - b^2(a^2 - b^2)r^2 - a^2(a^4 + b^4)}{(a^2 - b^2)}} \right),$$

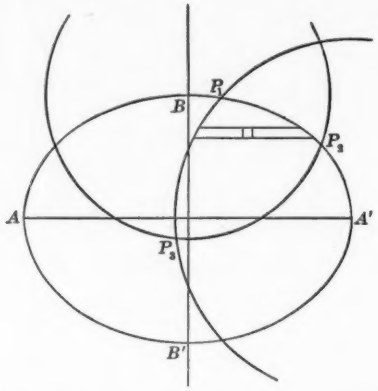
The area common to the ellipse and the circle is

$$\text{area} = 2\int_0^{y_1}\int_{a-\sqrt{r^2-y^2}}^{\frac{a}{b}\sqrt{b^2-y^2}} dx\,dy$$

$$= 2\left[\frac{a}{b}\frac{y_1}{2}\sqrt{b^2 - y_1^2} + \frac{b^2}{2}\sin^{-1}\frac{y'}{b} - ay' + \frac{y'}{2}\sqrt{r^2 - y_1^2} + \frac{r^2}{2}\sin^{-1}\frac{y_1}{r}\right],$$

which, by the conditions of the problem, $= \frac{1}{2}\pi ab$.

Substituting the value of y_1 in this equation, we have an equation in r. At the cost of great labor, this equation can be solved, by trial, to any degree of accuracy.



In a similar manner, the area over which a cow tethered at B can graze, may be found. Let the coördinates of the points of intersection, P_2, be (x_2, y_2). The length of the radius, BP_2, may be found to any desired degree of accuracy, as above. When this is found x_2 and y_2 become known.

Then to find the area common to the two circles and the ellipse we have only to perform the following integration,

$$\int_{y_3}^{y_2}\int_{a-\sqrt{r_2^2-(y-b)^2}}^{\sqrt{r_2^2-(y-b)^2}} dxdy + \int_{y_2}^{y_1}\int_{a-\sqrt{r_1^2-y^2}}^{(a/b)\sqrt{b^2-y^2}} dxdy,$$

where y_3 is the ordinate of P_3.

The case when the circle whose center is B cuts the ellipse in four points, would require still more labor.

343. Proposed by C. N. SCHMALL, New York City.

Show that the envelope of the system of circles whose radii are the ordinates of an ellipse is a concentric ellipse having the same minor axis as the given ellipse.

SOLUTION BY I. A. BARRETT and F. C. REISLER, University of Chicago.

Choose the axes so that the equation of the ellipse is in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Since the radii of the circles are to be ordinates of the ellipse, we have

$$(X - x)^2 + Y^2 = \frac{b^2}{a^2}(a^2 - x^2), \tag{1}$$

where (X, Y) are running coördinates of the circle.

Differentiating (1) with respect to x, we have

$$X - x = \frac{b^2}{a^2}x. \tag{2}$$

Solving (1) and (2) simultaneously, we obtain

$$X^2\left[1 - \frac{a^2}{a^2+b^2}\right]^2 + Y^2 = b^2 - \frac{b^2a^2}{(a^2+b^2)^2}X^2.$$

Reducing, we have

$$\frac{X^2}{a^2+b^2} + \frac{Y^2}{b^2} = 1.$$

This is an ellipse with same minor axis as original ellipse.

Also solved by FRANK C. MOORE, H. C. FEEMSTER, and the PROPOSER.

MECHANICS.

281. Proposed by C. N. SCHMALL, New York City.

ABC is a triangle inscribed in a circle, center *O*, and *L*, *M*, *N*, are the centers of gravity of the sectors *AOB*, *BOC*, *COA*. Show that

$$\frac{AB}{OL} + \frac{BC}{OM} + \frac{CA}{ON} = 3\pi.$$

SOLUTION BY S. W. REAVES, University of Oklahoma.

The well-known formula for the center of gravity of a sector of a circle gives

$$OL = \frac{4}{3}\cdot\frac{r\sin\frac{1}{2}AOB}{\text{angle } AOB} = \frac{AB}{3\angle C}.$$

Hence $\frac{AB}{OL} = 3\angle C$. Similarly, $\frac{BC}{OM} = 3\angle A$, and $\frac{CA}{ON} = 3\angle B$. Adding,

$$\frac{AB}{OL} + \frac{BC}{OM} + \frac{CA}{ON} = 3(\angle A + \angle B + \angle C) = 3\pi.$$

Also solved by A. M. HARDING, CHARLES E. HORNE, P. PEÑALVER, B. LIBBY, ELMER SCHUYLER, WALTER C. EELLS, RICHARD MORRIS, H. C. FEEMSTER, J. B. SMITH, J. W. COLSON, F. C. REISLER, and I. A. BARRET.

282. Proposed by R. P. LOCHNER, Philadelphia, Pa.

A car weighing 10 tons (2,240 lbs. each) attains a speed of 15 miles an hour from rest in 24 seconds, during which it covers 100 yards. If the space-average of the resistances is 30 lbs. per ton, find the average horse-power used to drive the car. (MORLEY'S *Mechanics for Engineers*, p. 66.)

SOLUTION BY J. W. CLAWSON, Collegeville, Pa.

Force (lbs.) required if there were no friction $= \frac{m}{g}\cdot\frac{v}{t} = \frac{2240}{32}\times\frac{22}{24} = 641$ lbs. approximately.

Force (lbs.) required to overcome friction = 300 lbs. Total force acting is therefore 941 lbs.

$$\text{Power applied} = \frac{Fs}{t} = \frac{941 \times 300}{24} = 11{,}762 \text{ ft. lbs. per sec.} = 21 \text{ H.P. (approx.).}$$

283. Proposed by C. N. SCHMALL, New York City.

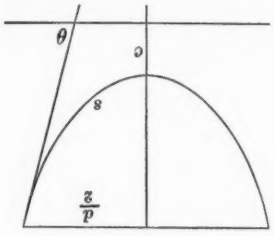
The maximum length of a certain chain which can be suspended from one end without breaking is l. It is desired to form a catenary with a length $2l/k$ of the chain, the points of support being a distance d apart, in the same horizontal line. Show that the maximum value of d is

$$\frac{2l}{k}(k^2-1)^{1/2}\log\left(\frac{k+1}{k-1}\right)^{1/2}.$$

SOLUTION BY A. M. HARDING, University of Arkansas.

Let w = weight per unit length. Then wl = maximum tension the chain will stand.

The tension at the point of support is given by $T\sin\theta = ws$ where s = one-half the length of the chain and θ is the angle that the tangent to the catenary at that point makes with the x-axis.



If we put $T = wl$ and $s = l/k$ we find $\sin\theta = 1/k$. But $\tan\theta = \frac{1}{2}(e^{d/2c} - e^{-(d/2c)})$ where c is the distance along the Y-axis from the origin to the catenary.

Hence we have

$$\tfrac{1}{2}(e^{d/2c} - e^{-(d/2c)}) = \frac{1}{\sqrt{k^2-1}}.$$

From this equation we obtain

$$d = 2c\log\left(\frac{k+1}{k-1}\right)^{\frac{1}{2}}.$$

We have also the intrinsic equation of the catenary $s = c\tan\theta$, from which we obtain

$$c = \frac{l}{k}\sqrt{k^2-1}.$$

Whence

$$d = \frac{2l}{k}(k^2-1)^{\frac{1}{2}}\log\left(\frac{k+1}{k-1}\right)^{\frac{1}{2}}.$$

Also solved by J. W. COLSON.

NUMBER THEORY.

189. Proposed by V. M. SPUNAR, Chicago, Illinois.

If p and $p_1 = 2^p - 1$ are primes, then are the numbers of the sequence $p_1 = 2^p - 1$, $p_2 = 2^{p_1} - 1$, $p_3 = 2^{p_2} - 1$, $\cdots$, $p_n = 2^{p_{n-1}} - 1$ all primes?

REMARKS BY R. D. CARMICHAEL, Indiana University.

If the conjecture stated in this problem is true it is highly desirable to have a proof of it. It would be a significant contribution to the theory of prime numbers; for the theorem so obtained would afford a means of recursion by which a sequence of indefinitely increasing prime numbers could be written out consecutively. No such tool has yet been invented. Euler and Legendre sought in vain for analytical expressions which would serve just this purpose. Fermat believed, though he confessed that he was unable to prove, that he had found such an analytical expression in

$$2^{2^n} + 1.$$

Euler pointed out the error of this opinion by showing that 641 is a factor of this number for the case $n = 5$.

From these historical facts one is led to suppose that the proof of the conjecture of this problem is probably difficult, in case the conjecture is true; if the conjecture is false, that would be shown by means of an example. The numbers to be dealt with, however, are so large that the construction of such an example (in case it exists) would perhaps be very tedious.

Note. A number of incorrect solutions of problem 198 have been received. Will our contributors please give it a more careful consideration? It is very probably not easy to solve.

EDITORS.

199. Proposed by R. P. LOCHNER, Philadelphia, Pa.

Find three integral squares, such that the sum of every two of them shall be a square.—Alsop's Algebra, 3d edition (Philadelphia, 1859), p. 296, Ex. 13.

SOLUTION BY ARTEMAS MARTIN, Washington, D. C.

Let x^2, y^2 and z^2 be the required squares; then we must have

$$x^2 + y^2 = \square,\ x^2 + z^2 = \square,\ y^2 + z^2 = \square, \qquad (1, 2, 3)$$

Assume $y = \dfrac{(m^2 - n^2)x}{2mn}$, $z = \dfrac{(p^2 - n^2)x}{2pn}$; then (1) and (2) are satisfied and (3) becomes, after striking out common square factors,

$$p^2(m^2 - n^2)^2 + m^2(p^2 - n^2)^2 = \square, \qquad (4)$$

which may be otherwise written

$$m^2p^2(m^2 + p^2) - 4m^2n^2p^2 + n^4(m^2 + p^2) = \square; \qquad (5)$$

and (5) will be a square when

$$m^2 + p^2 = 4n^2. \qquad (6)$$

Put $m = \dfrac{2n(r^2 - s^2)}{r^2 + s^2}$, $p = \dfrac{4nrs}{r^2 + s^2}$; then (6) is satisfied. Take $r = 2$, $s = 1$, and we have $m = \dfrac{6n}{5}$, $p = \dfrac{8n}{5}$; $y = \dfrac{11x}{60}$, $z = \dfrac{39x}{80}$. Now take $x = 240$ and we have $y = 44$, $z = 117$, the least numbers known.

The numbers given by Alsop (528, 5796, 6325) are much larger, and same as found by Bonnycastle in his Introduction to Algebra published in London more than 100 years ago.

For other solutions, see *Mathematical Magazine*, Vol. II, No. 11 (Dec., 1898), pp. 214–16; also, the *Normal Monthly*, Vol. III, No. 10 (Millersville, Pa., June, 1876), p. 119.

Also solved by C. E. GITHENS.

MISCELLANEOUS QUESTIONS.

EDITED BY R. D. CARMICHAEL.

QUESTION.

11. In our courses of study is it desirable to give more consideration to vector analysis? What topics should be included in a first treatment of this subject?

REPLY.

6. In what ways and to what extent will the teaching of mathematics and the content of the curriculum probably be affected by the increasing demand for vocational training?

I. REMARKS BY W. T. STRATTON, Kansas State Agricultural College, Manhattan, Kansas.

There is a demand today for the practical in education. Every subject that is to remain in our curriculum must do so by showing that it has a real place in the lives of the people. The demand for vocational training will affect both the teaching of mathematics and the content of the curriculum to a very marked degree.

Every new teacher in our high schools has a tendency to teach the topics that were taught to him, and in the same way as he was instructed. Only the unusual man will break away from this practice. So the change in teaching and the vocational attitude toward the subject must find a place in the colleges and universities before they will permeate into all the schools. As long as the former turn out men and women grounded in the belief that the chief reason for the study of mathematics is in its disciplinary value for the intellect the high schools will in general follow the old traditional course. The change in teaching, it seems to me, will come principally from a change of attitude of the teachers toward the subject. The teachers will come to look upon the subject from the social point of view, that is, they will not only teach the mathematical computations and methods of argumentation, but they will also give the subject a social setting; they will show to the students the place in society at which the particular topic under consideration has its direct applications.

The kind of training college men and women have had in the past has not fitted them for work in the vocational schools. Until the normal schools and colleges give vocational subjects, and train the teachers from the vocational point of view, nothing but poorly taught vocational mathematics can be expected in the high schools. I do not mean that the class room instruction will be essentially changed, but that there will be a closer correlation of the mathematics

work with that of the other departments, and that the pupils will be brought into contact with real problems and not the mere statements of problems. Nor do I believe that it will ever be practicable in high school mathematics for teachers to take their classes into the laboratories or workshops and there work on a series of experiments so planned as to present the problems which the pupils will then be required to solve. It appears that there are many possibilities along this line that have never been realized, but it does not seem that one of the chief aims in mathematical instruction can best be served in this way. Mathematics consists of a system of knowledge which demands logical thinking for its mastery; and the number and variety of problems that will come up in this laboratory work will not be great enough to give the requisite experience in thinking.

The advent of vocational training into the secondary schools will probably bring about a closer study of the needs of the pupils. They will not be required to take a year and a half of algebra and a course in plane and solid geometry irrespective of the length of time they will likely spend in school or of the vocations for which they are preparing. Efforts are being put forth to make the courses such that if it becomes necessary for the pupils to leave school at the close of any term their work in mathematics will be of such a nature as to be of value to them. The amount and the character of the mathematics for students who are expecting to attend college will likely not be lessened. In the larger industrial centers the tendency will be to widen the scope of subjects, that is, along with the traditional algebra and geometry the most practical parts of trigonometry and a course in applied mathematics summarizing and vitalizing the subjects learned will make up the course. In the agricultural high schools the course in mathematics will likely be shortened and a course in applied arithmetic and courses in algebra and geometry, which emphasize the most practical phases of interest to the farmer, will replace the present courses. The course for the girls in the different schools will also probably be shortened, and the subjects of arithmetic, algebra, and geometry will be treated along with the applications of a kind that will be of most interest and value to them.

The high school in all its courses will probably give more consideration to the special needs of the pupils than to their preparation for college. The colleges will require fifteen units of work with very few limitations. Thus the small schools will be free to offer courses best fitted for the students in their particular localities and at the same time give the training requisite for admission to college.

NOTES AND NEWS.

UNDER THE DIRECTION OF FLORIAN CAJORI.

In H. Weber's *Kleines Lehrbuch der Algebra*, 1912, the same proof of the simplicity of the alternating group of degree n, $n \neq 4$, appears twice, on pages 215 and 373 respectively.

The lectures on recent advances in physics, delivered by Professor Volterra at Clark University in September, 1909, were published in the double number of the *Archiv der Mathematik und Physik* which appeared February 17, 1914.

It is proposed to erect a monument in memory of P. S. Laplace at his birthplace, Beaumont-en-Auge, France. The money to defray the expense is to be raised by international subscription. The firm, M. Gauthier-Villars, Paris, France, has offered to receive subscriptions for this purpose.

The part of the *Encyclopédie des Sciences Mathématiques* dated December 23, 1913, is devoted to Projective Geometry (143 pages) and to Configurations (17 pages). Among the new parts announced in press is the rest of the article on Calculus of Variations, and an article on the Developments of Hydrodynamics. Tribune publique 23 is inserted in the present part.

Florian Cajori, of Colorado College, will attend the Napier Tercentenary Celebration next July in Edinburgh, and will present a historical paper on that occasion. He will spend the summer in Great Britain and later travel on the Continent. In the fall of 1915 he will resume his work at Colorado College.

The January, 1914, issue of the *Irish Journal of Education* contains a further instalment of the Report of the National (American) Committee of Fifteen on Geometry Syllabus, embracing that part of the historical introduction which refers to England. The whole report is to be printed in the *Journal* by permission of the Committee.

Professor L. C. Karpinski, of the University of Michigan, held an Institute on Arithmetic before the Toledo, Ohio, Teachers' Association during the week of March 2–7, 1914. His lectures were mostly to the grade teachers, but one was given to the high school teachers. His underlying purpose was to emphasize the practical side of arithmetic.

The courses in mathematics to be given at the summer session of the University of Michigan, June 26 to August 21, 1914, include plane geometry, solid geometry, elementary algebra, trigonometry, college algebra, analytic geometry, geometry and algebra (teachers' course), calculus (two courses), theory of finance, insurance, and statistics, history of mathematics, differential equations, projective geometry, theory of functions, infinite series and products, the theory of potential, advanced calculus. The members of the regular staff offering these courses are professors BEMAN, MARKLEY, and FORD, assistant professors BRADSHAW and KARPINSKI, and instructors POOR, HILDEBRANDT, FORSYTH, and MILLER.

The following courses in mathematics will be offered during the Summer session of Dartmouth College, July 6 to August 15, 1914: In addition to elementary college courses,—by Professor PITCHER, principles of calculus; by Professor BILL, solid analytic geometry, advanced aspects of elementary geometry (the latter intended primarily for teachers); by Dr. MORGAN, higher geometry, the teaching of elementary algebra.

The University of Illinois announces a course of lectures on General Analysis by Professor Fréchet of the University of Poitiers, France. The course will

consist of four lectures per week throughout both semesters of 1914–'15. Professor Fréchet is recognized as one of the leaders in this field of mathematics and through his publications has contributed largely to its literature.

The Illinois chapter of the Society of Sigma Xi held a regular meeting on April 15 at which Mr. A. O. MacGillivray spoke on "Spiders and their spinning work."

The Spring quarter gathering of the Society of Sigma Xi at the University of Chicago was held on Tuesday, May 19. The after-dinner speaker was Dr. Ira Remsen, President *Emeritus* of Johns Hopkins University, whose subject was "Reminiscences of Sylvester and Rowland."

The Harvard University Press announces the early publication of "The Hyperbolic Functions of Complex Variables" by Professor A. E. Kennelly.

Professor David Eugene Smith, of the Teachers College, Columbia University, will attend the Napier tercentenary celebration at Edinburgh, next July, and will present a historical paper on that occasion.

Professor H. Andoyer, of the University of Paris, announces that he has completed for somewhat later publication the calculation of tables giving to fifteen decimal places the natural values of the sines, tangents, and secants of angles up to 90 degrees, in steps of ten seconds.

At Columbia University leaves of absence have been granted to Professor T. S. Fiske for the first half year and to Professor H. B. Mitchell for the second half year of 1914–1915.

At a recent meeting of the New York section of the Association of the Teachers of Mathematics of the Middle States and Maryland, addresses were delivered by Professor C. J. Keyser on "The human worth of rigorous thinking" and by Professor H. E. Hawkes on "How can we teach our pupils in geometry to think?"

The Fifty-Second Annual Convention of the National Education Association will be held in St. Paul, Minn., July 4–11, 1914. The meetings of the National Council will begin on Saturday, July 4. Educational Sunday will be observed on July 5, and the general sessions will open on July 6.

In *L'Enseignement Mathematique* for January, 1914, N. Gennimatas proves that a Heronian triangle, that is a triangle such that the area is a rational function of the sides, is always similar to a triangle whose sides may be written in the form: $a = x^2 + y^2$, $b = (1 + y^2)x$, $c = (1 + x)(y^2 - x)$, the area of the latter being equal to cxy.

"Memorabilia Mathematica or the Philomath's Quotation Book" is a volume by Dr. R. E. Moritz of the University of Washington, just published by the Macmillan Company. The following topics are discussed in the twenty-one chapters: definitions and object of mathematics, the nature of mathematics, estimates of mathematics, the value of mathematics, the teaching of mathematics, study and research in mathematics, modern mathematics, the mathematician,

mathematics as a fine art, mathematics as a language, mathematics and logic, arithmetic, algebra, geometry, the calculus and allied topics, the fundamental concepts of time and space, paradoxes and curiosities.

In *Science* for April 24, 1914, Professor C. N. Moore, of the University of Cincinnati, calls attention to an address by Professor Thorndike, of Columbia University, before the Cincinnati Schoolmasters' Club in which some sweeping statements in regard to the educational value of mathematics are made. As against these statements, Professor Moore recalls the article of Professor Keyser, head of the department of mathematics at Columbia University, published in *Science* for December 5, 1913, on "The human worth of rigorous thinking." It is well worth while for all friends of mathematics to consider carefully the claims set forth by both of these writers.

On April 14, 1914, there was celebrated at the University of Palermo the thirtieth anniversary of the foundation of the Circolo Matematico of Palermo. On that occasion a gold medal was presented to G. B. Guccia, the founder of the society and of its organ, the *Rendiconti*. As the society is international, the co-operation of mathematicians of all countries was invited. The local committee of arrangements consisted of Professors M. Albeggiani, F. Raffaele, G. Bagnera, G. Pitrè, A. Venturi, M. Gebbia and M. de Franchis.

The interest taken in Italy in the four hundredth anniversary of the birth of Tartaglia is made evident by the notices appearing in the daily press. *La Tribuna* in Rome printed on February 9 a long notice of Tartaglia, written by the engineer V. Tonni-Bazza, who has made a special study of the life and works of the sixteenth century algebraist. Antonio Favaro of the University of Padua has published several articles about Tartaglia in technical journals.

The February issue of *Isis*, the new journal on the history of science, edited by George Sarton and published in Belgium, completes the first volume. It is a substantial volume of 824 pages. The last number contains the following articles on mathematics: "Les tendances actuelles de l'histoire des mathématiques" by the editor, "Le glorie matematiche della Granbretagna" by Gino Loria, "The origin of Cauchy's conceptions of a definite integral and the continuity of a function" by Philip E. B. Jourdain. The international character of the journal is indicated by the fact that the three contributors just named have written in French, Italian and English. A fourth original article is in German, by Ernst Bloch, on "Die chemischen Theorien bei Descartes und den Cartesianern."

The twenty-fifth anniversary of the founding of Teachers College, Columbia University, was celebrated February 21. At the end of the present academic year, the college will place its work, with but one exception, on a graduate basis. According to the *Columbia University Quarterly*, "the normal schools supply the rank and file of teachers of the common schools; the state universities, through their departments of education, are equipping the rank and file of the teachers of the secondary schools. There remains however the preparation of the super-

visors and administrators in these fields and the equipping of the teaching staffs of all these professional faculties. It is to these fields, then, that Teachers College prepares to confine itself . . . ."

Beginning with January 1, 1914, the *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht* is being edited by H. Schotten, W. Lietzmann and E. Grimsehl. Heretofore it was edited by H. Schotten alone. B. G. Teubner continues to publish this journal. In the third number of Vol. 45, W. Lietzmann makes a report on the teaching of the differential and integral calculus in oberrealschulen, realgymnasien and gymnasien at the present time. The calculus is taught in most of these schools, but is developed with less rigor than in university courses. The use of the derivative predominates over that of the differential. There is little fusion of the differential and integral calculus, the latter being taken up only in part or neglected altogether. The introduction of the calculus into these schools is generally looked upon as a step forward, in spite of the lack of coördination of the courses with the university courses. As Klein says, the present is a period of transition. One of the vital questions is the degree of rigor with which the subject should be developed in these schools.

The earlier and wider introduction of the calculus receives attention also in England. In the December, 1913, number of the *Mathematical Gazette*, C. S. Jackson gives the first instalment of a paper, entitled "The Calculus as an item in school mathematics," which he read before the London branch of the Mathematical Association. Twenty-five years ago only a few of the best students took up calculus; skill in manipulation was overemphasized, applications were neglected. More recent demands made by physicists and engineers have led to a cleavage in the modes of initiating boys into the calculus. "The rigorist puts minute logical precision first—the practical man asks us to dwell on the utility of the calculus. Of the two the rigorist, in relation to the schoolboy, is by far the more astray. . . . We as teachers are just learning that you cannot teach people to generalize by throwing ready-made generalizations at them—that a grip on the concrete facts must precede a critical analysis of them—that there is no worse mistake than to jump before you come to the fence." Jackson's aim is expressed by himself as follows: "It is the object of this paper to argue that in broad outline there is one way—one fairly definite sequence—to which the teaching of the subject ought with variations to conform." In the part of his paper thus far published he enters upon a description of the preliminary work.

The following courses in mathematics will be offered at the Summer session of the University of Chicago, June 15 to August 28, 1914: By Professor G. A. Bliss, plane analytic geometry, and metric differential geometry; by Professor D. R. Curtiss, differential calculus, and functions of a complex variable; by Professor L. E. Dickson, theory of equations, theory of invariants (1st term), and linear algebra (2d term); by Professor Kurt Laves, analytic mechanics, and observatory practice; by Professor A. C. Lunn, geometrical optics, and electron theory of electromagnetism; by Professor C. N. Moore, college algebra,

and integral calculus; by Professor E. H. MOORE, solid analytic geometry (1st term), and integral equations in general analysis (1st term); by Professor F. R. MOULTON, problem of three bodies; by Professor G. W. MYERS, teaching of elementary school mathematics, and teaching of secondary school mathematics; by Professor H. E. SLAUGHT, plane trigonometry, and elliptic integrals; by Professor J. W. A. YOUNG, college geometry, and critical review of secondary mathematics. Also private reading and research in both pure and applied mathematics is conducted for advanced students. The mathematics club, consisting of all instructors and advanced students in mathematics and mathematical astronomy, meets weekly for discussion of research and pedagogic questions.

At the recent meeting of the Ohio Association of Mathematics and Science Teachers, a number of teachers of mathematics discussed the advisability of forming a mathematics section under the Ohio Academy of Sciences. Another suggestion was that the group of teachers concerned petition the council of the American Mathematical Society, of which many of them are members, for permission to organize a Junior Section of the Society for the purpose of conducting meetings at which papers of less formal character and of more direct bearing on mathematical teaching than is usual at regular meetings of the Society could be presented and discussed. Such meetings would be of value both to those who find it too expensive and inconvenient to attend regular meetings of the Society as now constituted, and to those whose interests lie chiefly in the field of mathematical teaching. It was further suggested that notices and reports might be published in the AMERICAN MATHEMATICAL MONTHLY, and that this journal might be made an official organ of such junior sections in case the plan were eventually recognized and approved by the Society.

Erratum.—In Professor Roever's article in the December, 1913, issue, the last sentence at the bottom of page 303 should be modified to read as follows: "This is an element in which a plane through the vertex and perpendicular to the internal bisector of the angle P_1PP_2 (Fig. 3) cuts the conical roof."

6

THE

AMERICAN

MATHEMATICAL MONTHLY

VOLUME XXI JUNE, 1914 NUMBER 6

THE CONSTRUCTION OF CONICS UNDER GIVEN CONDITIONS.

By B. M. WOODS, University of California.

The following discussion treats by synthetic methods the matter presented by Professor M. W. Haskell[1] in a paper of the same title, where the treatment is for the most part analytic.

SECTION I. CONICS DEFINED BY FIVE POINTS OR BY FIVE TANGENTS.

Theorem I. *The eight vertices of two quadrangles with the same diagonal triangle lie on a conic.*

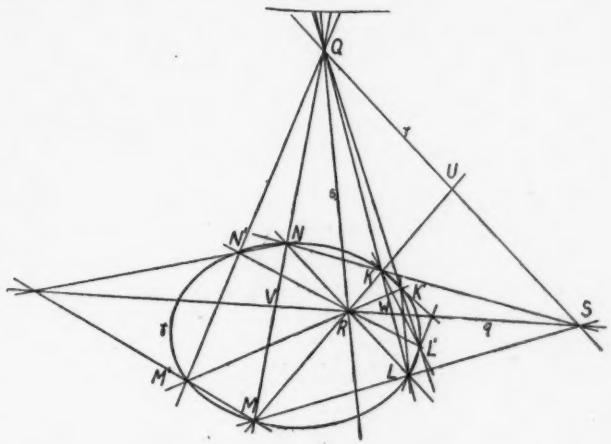


Fig. 1.

Consider a point Q (Fig. 1) and its polar q with respect to a given conic γ. If a point R is selected on q, its polar r passes through Q and cuts q in a point S, whose polar s is the line QR. A self-polar triangle QRS is thus determined.

[1] *Bulletin of the Amer. Math. Soc.*, 2d Series, Vol. XI, No. 5, pp. 268–273.